CALCULATION OF THE INVARIANT MANIFOLDS OF THE LIE GROUP USING ITS DEFINING EQUATIONS

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A wide class of mechanical problems lends itself to the group-theoretic treatment. A connection between the integrals of differential equations and the invariants of continuous groups, was first recognized by Lie [1]. A theorem by Noether [1 and 2] showed the connection between the conservation laws and the property of invariance of the action function. Kinematics of the rigid body can be treated as a theory of invariants of a group of motions supplemented with the condition of invariance of time. Group-theoretic treatment can also be applied to the Hamiltonian systems, which form a set of differential invariant manifolds of the group of contact transformations.

In the problem of optimal stabilization, the control function can be regarded as an invariant of a continuous group of transformations preserving the equations of the stabilizable system and the corresponding variational Euler's equations. Such a group classifies the initial values of solutions of the system, according to the closeness of these solutions (in the limit) to the trivial solution.

Special theory of relativity can be treated as a theory of invariants of a group (Lorentz') preserving the plane, pseudo-Euclidean space-time metric (see e.g. [3] for the group-theoretic treatment of physics).

Invariants of the group must also be attained when the group-theoretic approach is used to find the solutions of equations of mathematical physics [4 and 5] and it is in connection with the group-theoretic interpretations of such applied problems that we analyze the feasibility of constructing finite (nondifferential) invariant manifolds of the continuous group G directly from its defining equations (without integration).

We prove that any manifold at the points of which the defining equations degenerate (this means lowering the rank of the matrix of the left-hand side coefficients of the defining equations), is an invariant manifold of the group G. Equations of these manifolds can be written out explicitly. When the group G is intransitive, then its defining equations undergo the identity degeneration. To obtain the invariants of the group, we write explicitly a concurrent Pfaffian system whose rank is equal to the order of the group G. Necessary and sufficient conditions for the local transitivity of G are formulated for the case when any of its points is a general position point.

1. Statement of the problem. The idea of group-theoretic treatment of physical problems is attributed to Klein, who applied it to geometry [6]. However, as we said before, situations when certain facts can be formulated in terms of the invariants of some group of transformations, may be encountered outside the field of geometry.

The main difficulty arising here is as follows. As we know ([7]. pp. 184-211), any Lie group G on the space E_n of variables $x_1, \ldots, x_n$, can be defined using a linear system of partial differential equations, which we shall call, following Lie, the system of defining equations. These equations connect the components $\xi_i(x)$ of the infinitesimal operators $X = \xi_i(x)\, \partial/\partial x_i$ which themselves are the elements of the associated Lie algebra. Invariants and invariant manifolds of G or of its continuation, can be calculated using the well known procedures [7 and 8]. The latter can however be applied only after the functions $\xi_i(x)$ have been obtained by integrating the defining equations, and it is the integrating process that causes the main difficulty.

Let us consider the following system of defining equations of some group G

$$a_{\gamma i}^{\varepsilon}(x)\frac{\partial \xi_i}{\partial x_\varepsilon} = \alpha_{\gamma i}(x)\,\xi_i \qquad (i, \varepsilon = 1, \ldots, n;\ \gamma = 1, \ldots, m) \qquad (1.1)$$

where i and ε are the dummy summation indices.

We shall attempt to determine all nondifferential invariants of the subsets of the group G directly from (1.1), whenever it can be done without recourse to integration.

2. Reduction of the system of defining equations to passivity. Let us assume that (1.1) has solved for various derivatives of the unknown functions and, that neither these derivatives nor the second derivatives enter the right-hand sides of the defining equations. We shall call these derivatives principal, and all the remaining ones —parametric [9].

If, after all possible differentiations of (1.1) have been performed and the principal derivatives eliminated from the right-hand sides we find, that not a single pair of independent equations with identical left-hand side parts exists, it will mean that no connection exists between the parametric derivatives. In this case the system (1.1) shall be called passive.

If (1.1) is not passive, we can make it passive according to the Riquier theory [9] by supplementing it with a finite number of equations associated with (1.1) algebraically and differentially.

It can be shown that a finite number of steps is required to bring (1.1) to the passive state, or to show its incompatibility.

Indeed, each elementary step of supplementing (1.1) is based on adding a certain number of equations in accordance with the comparison formulas for the monomials assigned to the principal derivatives of the system (1.1). Since the new principal derivatives appearing during the differentiation of the parametric derivatives do not belong to the set of the principal derivatives of (1.1), therefore they cannot be obtained from the latter by differentiation.

Consequently, the monomials assigned to the new principal derivatives are not divisible by the monomials assigned to the principal derivatives of (1.1). We can say that the sequence $p(\alpha)$ of the vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative components does not increase, if for the vector α' following the vector α, at least one of the differences $\alpha_i' - \alpha_i$ is negative.

It therefore follows, that the sequence of monomials assigned to the new principal derivatives in each step, does not decrease. Hence by the Lemma of [9] (p. 68) the number of new principal derivatives and, consequently, the number of equations supplementing (1.1) to make it passive, is finite. The property of passivity of (1.1) will play an important part in the later stages.

3. Necessary conditions for the defining equations. We shall obtain the conditions which should be satisfied by the coefficients $a_{\gamma i}^{\varepsilon}$ of the system (1.1) for the case, when (1.1) forms a system of defining equations for some group.

Let us consider the matrix $\|a_{\gamma i}^{\varepsilon}\|$ where γ denotes the γth row and where the columns are ordered in the following manner. We shall set a correspondence between each pair of numbers (ε, i) and the column number

$$\gamma_1 = (\varepsilon - 1)n + i \tag{3.1}$$

Assuming that $\varepsilon \leqslant n$ and $i \leqslant n$, it is easy to see that the column number γ_1 defines ε and i uniquely. Let us denote the overall ranks of the matrices $\|a_{\gamma i}{}^{\varepsilon}\|$ and $\|\alpha_{\gamma i}\|$ by r and h respectively. Obviously

$$0 \leqslant m \leqslant n^2 + n,\ 0 \leqslant h \leqslant \min(m, n),\ 0 \leqslant r \leqslant \min(m, n^2) \tag{3.2}$$

Let $\xi'_1, \ldots, \xi'_n$ and $\xi_1, \ldots, \xi_n$ be two solutions of (1.1) and $X' = \xi_i'\partial/\partial x_i$ with $X = \xi_i\partial/\partial x_i$ the corresponding infinitesimal operators. Then by the Lie theorem [7] the functions $X'\xi_1 - X\xi'_1, \ldots, X'\xi_n - X\xi_n'$ should also be a solution of (1.1). We have

$$0 = a_{\gamma i}{}^k \frac{\partial}{\partial x_k}(X'\xi_i - X\xi_i') - \alpha_{\gamma i}(X'\xi_i - X\xi_i') =$$

$$= \left(a_{\gamma i}{}^k \frac{\partial \xi_i}{\partial x_l} - a_{\gamma l}{}^i \frac{\partial \xi_k}{\partial x_i} + \xi_i \frac{\partial a_{\gamma l}{}^k}{\partial x_i}\right)\frac{\partial \xi_l'}{\partial x_k} + \left(\xi_i \frac{\partial \alpha_{\gamma i}}{\partial x_l} - \xi_i \frac{\partial \alpha_{\gamma l}}{\partial x_i} - \frac{\partial a_{\gamma i}{}^k}{\partial x_l}\frac{\partial \xi_i}{\partial x_k}\right)\xi_l' +$$

$$+ X'\left(a_{\gamma i}{}^k \frac{\partial \xi_i}{\partial x_k} - \alpha_{\gamma i}\xi_i\right) - X\left(a_{\gamma i}{}^k \frac{\partial \xi_i'}{\partial x_k} - \alpha_{\gamma i}\xi_i'\right) \tag{3.3}$$

and by (1.1),

$$X'\left(a_{\gamma i}{}^k \frac{\partial \xi_i}{\partial x_k} - \alpha_{\gamma i}\xi_i\right) = X\left(a_{\gamma i}{}^k \frac{\partial \xi_i'}{\partial x_k} - \alpha_{\gamma i}\xi_i'\right) = 0$$

Let now the system (1.1) be such, that none of the supplementing equations is of the first order. Then each of the equations of (3.3) should be an algebraic consequence of (1.1). Therefore such regular functions $\lambda_{\gamma i}$ can be found, that, in addition to (1.1), the relations

$$\begin{gathered}(a_{ji}{}^k\delta_l{}^\varepsilon - a_{jl}{}^\varepsilon\delta_i{}^k)\frac{\partial \xi_l}{\partial x_\varepsilon} + \frac{\partial a_{jl}{}^k}{\partial x_i}\xi_l = \lambda_{j\gamma}a_{\gamma l}{}^k \\ \frac{\partial a_{ji}{}^\varepsilon}{\partial x_l}\frac{\partial \xi_i}{\partial x_\varepsilon} + \left(\frac{\partial \alpha_{jl}}{\partial x_i} - \frac{\partial \alpha_{ji}}{\partial x_l}\right)\xi_l = \lambda_{j\gamma}\alpha_{\gamma l} \\ (k, l, \varepsilon, i = 1, \ldots, n;\quad j, \gamma = 1, \ldots, m)\end{gathered} \tag{3.4}$$

where $\delta^\varepsilon{}_l$ denotes the Kronecker delta, will also be fulfilled.

The relations $0 \leqslant h \leqslant m \leqslant n^2$ also hold for h, m and n.

We shall first consider the case $0 < h = m \leqslant n$. Linear combinations of (1.1) yield

$$a_{jl}{}^\varepsilon \frac{\partial \xi_l}{\partial x_\varepsilon} = \xi_j + \alpha_{j\omega}\xi_\omega \qquad \left(\begin{matrix}\varepsilon, i = 1, \ldots, n;\ j = 1, \ldots, m \\ \omega = m+1, \ldots, n\end{matrix}\right) \tag{3.5}$$

while the conditions (3.4) become

$$(a_{jl}{}^k\delta_l{}^\varepsilon - a_{jl}{}^\varepsilon\delta_i{}^k)\frac{\partial \xi_l}{\partial x_\varepsilon} + \frac{\partial a_{jl}{}^k}{\partial x_i}\xi_i = \lambda_{j\gamma}a_{\gamma l}{}^k \tag{3.6}$$

$$\frac{\partial a_{ji}{}^\varepsilon}{\partial x_\lambda}\frac{\partial \xi_l}{\partial x_\varepsilon} - \frac{\partial \alpha_{j\omega}}{\partial x_\lambda}\xi_\omega = \lambda_{j\lambda} \qquad (\lambda = 1, \ldots, m) \tag{3.7}$$

$$\frac{\partial a_{ji}{}^\varepsilon}{\partial x_\rho}\frac{\partial \xi_l}{\partial x_\varepsilon} - \frac{\partial \alpha_{j\omega}}{\partial x_\rho}\xi_\omega + \frac{\partial \alpha_{j\rho}}{\partial x_\lambda}\xi_\lambda + \frac{\partial \alpha_{j\rho}}{\partial x_\omega}\xi_\omega = \lambda_{j\gamma}\alpha_{\gamma\rho} \tag{3.8}$$

$$(\rho = m+1, \ldots, n)$$

Inserting into (3.6) and (3.8) the values of $\lambda_{j\lambda}$ from (3.7) and ξ_j from (3.5), we obtain equations which should be identities in $\partial\xi_i/\partial x_\varepsilon$ and ξ_ω taken as independent variables. This yields the following conditions:

$$a_{ji}^{k}\delta_{l}^{\varepsilon}-a_{jl}^{\varepsilon}\delta_{i}^{k}+a_{\gamma i}^{\varepsilon}\frac{\partial a_{jl}^{k}}{\partial x_{\gamma}}-a_{\gamma l}^{k}\frac{\partial a_{ji}^{\varepsilon}}{\partial x_{\gamma}}=0$$

$$\frac{\partial a_{ji}^{\varepsilon}}{\partial x_{\rho}}+a_{\lambda i}^{\varepsilon}\frac{\partial \alpha_{j\rho}}{\partial x_{\lambda}}-\alpha_{\gamma\rho}\frac{\partial a_{ji}^{\varepsilon}}{\partial x_{\gamma}}=0$$

$$\frac{\partial \alpha_{j\rho}}{\partial x_{\omega}}-\frac{\partial \alpha_{j\omega}}{\partial x_{\rho}}+\alpha_{\gamma\rho}\frac{\partial \alpha_{i\omega}}{\partial x_{\gamma}}-\alpha_{\gamma\omega}\frac{\partial \alpha_{j\rho}}{\partial x_{\gamma}}=0$$

Let us now define the operators

$$Y_{j}^{\varepsilon}=a_{\gamma i}^{\varepsilon}\frac{\partial}{\partial x_{\gamma}},\qquad Z_{\omega}=-\frac{\partial}{\partial x_{\omega}}+\alpha_{\gamma\omega}\frac{\partial}{\partial x_{\gamma}}$$

Then the above conditions will respectively become

$$(Y_{i}^{\varepsilon},Y_{i_1}^{\varepsilon_1})=\delta_{i}^{\varepsilon_1}Y_{i_1}^{\varepsilon}-\delta_{i_1}^{\varepsilon}Y_{i}^{\varepsilon_1},\qquad (Y_{i}^{\varepsilon},Z_{\omega})=0,\qquad (Z_{\omega},Z_{\rho})=0 \quad (3.9)$$

Let us put $0 \leqslant h < m < n^2$. Combining the Eqs. (1.1) linearly, we can write

$$a_{ji}^{\varepsilon}\frac{\partial \xi_{l}}{\partial x_{\varepsilon}}=\xi_{j}+\alpha_{j\omega}\xi_{\omega},\qquad a_{\rho i}^{\varepsilon}\frac{\partial \xi_{l}}{\partial x_{\varepsilon}}=0 \qquad (3.10)$$

$$j=1,\ldots,h;\quad i,\varepsilon=1,\ldots,n;\quad \omega=h+1,\ldots,n;\quad \rho=h+1,\ldots,n$$

Let us solve, at the general points, the second system of (3.10) for various derivatives and insert the results into the first system. This, by (3.1), will define the set L consisting of ($m-n$) pairs of such numbers (α,β), that

$$a_{\gamma l}^{k}=\delta_{\gamma_0}^{\gamma}\ \text{when}\ \gamma>n\ \ (k,l)\in L;\qquad a_{\gamma l}^{k}=0\ \text{when}\ \gamma\leqslant n\ \ (k,l)\in L$$

$$\gamma_0=(\alpha-1)n+\beta,\qquad \gamma=(k-1)n+l$$

With the defining equations written in this form, we can obtain the following expressions for some of the conditions (3.4)

$$\lambda_{\rho j}=\frac{\partial a_{\rho i}^{\varepsilon}}{\partial x_{j}}\frac{\partial \xi_{l}}{\partial x_{\varepsilon}},\qquad \lambda_{j\varkappa}=\frac{\partial a_{ji}^{\varepsilon}}{\partial x_{\varkappa}}\frac{\partial \xi_{l}}{\partial x_{\varepsilon}}-\frac{\partial \alpha_{ji}}{\partial x_{\varkappa}}\xi_{l}$$

$$\lambda_{j\sigma}=(a_{ji}^{\alpha}\delta_{\beta}^{\varepsilon}-a_{j\beta}^{\varepsilon}\delta_{i}^{\alpha})\frac{\partial \xi_{i}}{\partial x_{\varepsilon}},\qquad \lambda_{\sigma\sigma'}=(a_{\sigma i}^{\alpha'}\delta_{\beta'}^{\varepsilon}-a_{\sigma\beta'}^{\varepsilon}\delta_{l}^{\alpha'})\frac{\partial \xi_{l}}{\partial x_{\varepsilon}}$$

$$(\sigma,\sigma'=h+1,\ldots,m;\ \varkappa=1,\ldots,h)$$

$$\sigma=(\alpha-1)n+\beta,\quad \sigma'=(\alpha'-1)n+\beta,\quad (\alpha,\beta)\in L,\quad (\alpha',\beta')\in L$$

Inserting into the remaining equations of (3.4) the expressions obtained for λ and the values of ξ_j from (3.10) we obtain a system of equations, each of which should be identically satistied in ξ_ω and be an algebraic consequence of the second system of (3.10). The condition that the coefficents of ξ_ω in these equations vanish identically, yields

$$\frac{\partial \alpha_{j\omega}}{\partial x_{\mu}}-\frac{\partial \alpha_{j\mu}}{\partial x_{\omega}}+\alpha_{\varkappa\omega}\frac{\partial \alpha_{j\mu}}{\partial x_{\varkappa}}-\alpha_{\varkappa\mu}\frac{\partial \alpha_{j\omega}}{\partial x_{\varkappa}}=0,\quad \mu=h+1,\ldots,n \qquad (3.11)$$

$$\frac{\partial a_{jl}^{k}}{\partial x_{\mu}}+a_{\varkappa l}^{k}\frac{\partial \alpha_{j\mu}}{\partial x_{\varkappa}}-\alpha_{\varkappa\mu}\frac{\partial a_{jl}^{k}}{\partial x_{\varkappa}}=0,\qquad \frac{\partial a_{\rho l}^{k}}{\partial x_{\mu}}-\alpha_{\varkappa\mu}\frac{\partial a_{\rho l}^{k}}{\partial x_{\varkappa}}=0$$

$$\psi_{\gamma il}^{\varepsilon k}\frac{\partial \xi_{l}}{\partial x_{\varepsilon}}=0\qquad \gamma=1,\ldots,m$$

Those of the above equations in which

$$\psi_{\gamma i l}^{\varepsilon k} = a_{\varkappa i}^{\varepsilon} \frac{\partial a_{\gamma l}^{k}}{\partial x_{\varkappa}} - a_{\varkappa l}^{k} \frac{\partial a_{\gamma i}^{\varepsilon}}{\partial x_{\varkappa}} + a_{\gamma i}^{k} \delta_{l}^{\varepsilon} - a_{\alpha l}^{\varepsilon} \delta_{i}^{k} - a_{\sigma l}^{k} (a_{\gamma i}^{\alpha} \delta_{\beta}^{\varepsilon} - a_{\gamma \beta}^{\varepsilon} \delta_{i}^{\alpha})$$

should be an algebraic consequence of

$$a_{\rho i}{}^{\varepsilon} (\partial \xi_i / \partial x_{\varepsilon}) = 0$$

Therefore regular functions $\lambda_{\gamma l}^{kp}$ exist such, that the relations

$$\psi_{\gamma i l}^{\varepsilon k} \frac{\partial \xi_l}{\partial x_{\varepsilon}} = \mu_{\gamma l}^{k\rho} a_{\rho i}^{\varepsilon} \frac{\partial \xi_i}{\partial x_{\varepsilon}}$$

will be identities in $\partial \xi_i / \partial x_{\varepsilon}$ taken as independent variables and $\psi_{\gamma i l}^{\varepsilon k} = \mu_{\gamma l}^{k\rho} a_{\rho i}^{\varepsilon}$. The last equations yield

$$\mu_{\gamma l}^{k\sigma'} = \psi_{\gamma \beta' l}^{\alpha' k}, \quad \sigma' = (\alpha' - 1) n + \beta', \quad (\alpha', \beta') \in L$$

$$\psi_{\gamma i l}^{\varepsilon k} = a_{\sigma' i}^{\varepsilon} \psi_{\gamma \beta' l}^{\alpha' k} \tag{3.12}$$

where

$$\psi_{\gamma \beta' l}^{\alpha' k} = a_{\gamma \beta'}^{k} \delta_{l}^{\alpha'} - a_{\gamma l}^{\alpha'} \delta_{\beta'}^{k} - a_{\sigma l}^{k} (a_{\gamma \beta'}^{\alpha} \delta_{\beta}^{\alpha'} - a_{\gamma \beta}^{\alpha'} \delta_{\beta'}^{\alpha})$$

Let us now introduce additional variables $y_{n+1}, \ldots, y_m$ and define the operators

$$Y_l{}^k = a_{\varkappa l}^{k} \frac{\partial}{\partial x_{\varkappa}} + a_{\rho l}{}^{k} \frac{\partial}{\partial y_{\rho}}, \quad Z_{\omega} = -\frac{\partial}{\partial x_{\omega}} + \alpha_{\varkappa \omega} \frac{\partial}{\partial x_{\omega}}$$

Then the necessary conditions for the defining equations (3.11) and (3.12) can be written in the form $(Z_{\mu}, Z_{\omega}) = 0$, $(Z_{\mu}, Y_l{}^k) = 0$ and

$$(Y_i{}^{\varepsilon}, Y_l{}^{k}) = \delta_i{}^k Y_l{}^{\varepsilon} - \delta_l{}^{\varepsilon} Y_i{}^k + a_{\sigma l}{}^k (\delta_{\beta}{}^{\varepsilon} Y_i{}^{\alpha} - \delta_i{}^{\alpha} Y_{\beta}{}^{\varepsilon}) +$$

$$+ a_{\sigma' i}^{\varepsilon} (\delta_l{}^{\alpha'} Y_{\beta'}^{k} - \delta_{\beta'}^{k} Y_l{}^{\alpha'}) - a_{\sigma' i}^{\varepsilon} a_{\sigma l}^{k} (\delta_{\beta}{}^{\alpha'} Y_{\beta'}{}^{\alpha} - \delta_{\beta'}{}^{\alpha} Y_{\beta}{}^{\alpha'}) \tag{3.13}$$

In particular, when $a_{h+1,i}^{\varepsilon} = \ldots = a_{mi}{}^{\varepsilon} = 0$, then Eqs. (3.13) yield (3.9).

4. Theorem of the invariant manifolds. When $h = 0$, Eqs. (1.1) admit a transitive group of transformations $\xi_j = \delta_j{}^i$; $i = 1, \ldots, n$.

Let now $0 < h \leqslant m < n^2$. As stated in the previous Section, the system (1.1) can be transformed into (3.10). In the following we shall assume, that:

1) a finite region Γ' of the space $\{x\}$ exists, in which the functions $a_{1i}{}^{\varepsilon}, \ldots, a_{hi}{}^{\varepsilon}$ and $\alpha_{j\omega}$ are regular and

2) the quantities $a_{h+1,i}^{\varepsilon}, \ldots, a_{mi}^{\varepsilon}$ are real constants (this constraint is not easily removable, since the procedure of making the defining equations passive is not complete).

Then, from the conditions (3.13) it follows that the operators $Y_l{}^k$ and Z_{ω} formed with the help of the coefficients of the defining equations (3.10) will be the elements of the $(n^2 + n - h)$-dimensional algebra L of the infinitesimal operators. Numbers $a_{h+1,i}^{\varepsilon}, \ldots, a_{mi}{}^{\varepsilon}$ will be subject to constraints following from the conditions for the structural constants. We shall denote the group corresponding to this algebra, by G^*. The matrix of L will be of the order of $(n^2 + n - h) \times (m + n - h)$ and will have the form

$$\begin{Vmatrix} a_{ji}{}^{\varepsilon} & 0 & a_{\rho i}{}^{\varepsilon} \\ \alpha_{j\omega} & -E & 0 \end{Vmatrix} \tag{4.1}$$

where E denotes a $(n - h)$-th order unit square matrix. The general rank of (4.1) will be $q \leqslant m + n - h$. If the rank of (4.1) is not reduced at any point of Γ', then, by the Riquier [9] theorem a group G exists, which is continuous and transitive at any point of the region $\Gamma \subset \Gamma'$. Using the well known facts of the theory of invariants of the Lie groups [7 and 8] we shall first show that, if the solutions of (3.10) are continuous in $\Gamma \subset \Gamma'$, then any invariant manifold I of the group G^* belonging to Γ will be an invariant manifold of G. Let us consider an arbitrary point $x^\circ = (x^\circ_1,\ldots, x^\circ_n) \in \Gamma$. Let the rank of the matrix (4.1) be lowered at this point by t units relative to q. If $t \neq 0$, then x° belongs to a sequence of invariant manifolds of the group G^*, contained within one another. Suppose that the smallest of these manifolds I_p is of dimension p and is defined by $n - p$ independent equations

$$\varphi_1(x) = \ldots = \varphi_{n-p} \quad (x) = 0 \tag{4.2}$$

which are obtained from the condition that all minors of the order of $t+1$ of the matrix (4.1), vanish. The following relations will hold at x° and at all points of I_p

$$Y_i^{\varepsilon}\varphi_s = a_{\varkappa i}^{\varepsilon} \frac{\partial \varphi_s}{\partial x_\varkappa} = 0 \qquad (s = 1, \ldots, n-p) \tag{4.3}$$

$$Z_\omega \varphi_s = -\frac{\partial \varphi_s}{\partial x_\omega} + \alpha_{\varkappa\omega} \frac{\partial \varphi_s}{\partial x_\varkappa} = 0 \tag{4.4}$$

Multiplying all jth equations of the first system of (3.10) by $\partial\varphi_s/\partial x_j$ and adding, we obtain, by (4.3),

$$\xi_j \frac{\partial \varphi_s}{\partial x_j} + \frac{\partial \varphi_s}{\partial x_j} \alpha_{j\omega}\xi_\omega = 0$$

If G is continuous in Γ, then the above relations should hold along I_p. Taking (4.4) into account we obtain

$$0 = \xi_j \frac{\partial \varphi_s}{\partial x_j} + \frac{\partial \varphi_s}{\partial x_j}\alpha_{j\omega}\xi_\omega = \xi_j \frac{\partial \varphi_s}{\partial x_j} + \xi_\omega \frac{\partial \varphi_s}{\partial x_\omega} = \xi_i \frac{\partial \varphi_s}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{4.5}$$

which shows that the set I_p is an invariant manifold of G.

From (4.3) it follows that $t \geqslant n - p$. If $t = n - p$, then the manifold I_p will be the only invariant manifold of G of dimension p, containing the point x°. Indeed, in this case the conditions connecting the magnitudes ξ_i at the point x° require exactly $n - p$ relations (4.5) and the group G is therefore transitive near the point x° belonging to I_p. If we had another invariant set $I_p' \neq I_p$ also passing through x°, then the intersection $I'_p \cap I_p$ would also be an invariant set of G of dimension less than p, and this would be impossible by virtue of the transitive property mentioned above. If $t > n - p$, then the group G^* exists on I_p intransitively and admits on it exactly $t + p - n$ invariants

$$\psi_1 = \text{const}, \ldots, \psi_{r-p\ n} = \text{const} \tag{4.6}$$

In this case, the following relations will hold, in addition to (4.3) and (4.4), at the points of I_p

$$Y_i^{\varepsilon}\psi_\nu = Z_\omega\psi_\nu = 0 \qquad (\nu = 1, \ldots, r + p - n) \tag{4.7}$$

and they will imply, as in the previous case, the conditions $\xi_i\, \partial\psi_\nu/\partial x_i = 0$ hold on I_p. This in turn implies, that the intersections of the surfaces (4.6) with I_p will be invariant manifolds of G of dimension $p - 1$ and we can therefore say that no other invariant manifolds of G of dimension $p - 1$ pass through x°.

The above arguments are valid for all $p = 0, \ldots, n$. Assuming lastly that every set I_p of the group contains all its invariant manifolds of dimension less than p, we obtain the following theorem.

T h e o r e m 4.1. (1) If the rank $m + n - h$ of the matrix (4.1) does not decrease at any point of the region Γ, then the group G is locally transitive at any point of this region. This condition is necessary.

2) Let the system (1.1) of the defining equations be reduced to the form (3.10)

$$a_{ji}^{\varepsilon}\frac{\partial \xi_i}{\partial x_\varepsilon} = \xi_j + \alpha_{j\omega}\xi_\omega, \qquad a_{\rho i}^{\varepsilon}\frac{\partial \xi_i}{\partial x_\varepsilon} = 0$$

$$(j = 1, \ldots, h;\ i, \varepsilon = 1, \ldots, n;\ \omega = h+1, \ldots, n;\ \rho = h+1, \ldots, m)$$

where a_{ij}^{ε} are functions of x regular in Γ' while $a_{\rho i}^{\varepsilon}$ are constants.

Then the infinitesimal operators

$$Y_i^{\varepsilon} = a_{ji}^{\varepsilon}\frac{\partial}{\partial x_j} + a_{\rho i}^{\varepsilon}\frac{\partial}{\partial y_\rho}, \qquad Z_\omega = -\frac{\partial}{\partial x_\omega} + \alpha_{j\omega}\frac{\partial}{\partial x_j}$$

generate an algebra with the matrix (4.1),

$$M = \begin{Vmatrix} a_{ji}^{\varepsilon} & 0 & a_{\rho i}^{\varepsilon} \\ \alpha_{j\omega} & -E & 0 \end{Vmatrix}$$

corresponding to some Lie group G^*. If Eqs. (3.10) admit solutions regular in $\Gamma \subset \Gamma'$, then the group G defined by the set of all such solutions admits, in Γ, all invariant manifolds of G^*. This exhausts all invariant manifolds of G of dimension $p \leqslant p_0$ where p_0 is the greatest dimension of the invariant manifolds of G^*. From this it follows, in particular, that if $p_0 = n - 1$, then the invariant manifolds of G^* yield all invariant manifolds of G.

3) If in (1.1) all $\alpha_{ji} = 0$ ($h = 0$), then the group G defined by these equations is locally transitive at all points and does not admit any invariant manifolds.

5. Notes on application of the Theorem 4.1. (1) Singular invariant manifolds. In accordance with a well known procedure, the invariant manifolds of G^* are calculated as follows. We equate to zero all minors of the order $n-h+1$ of the matrix M. The resulting system of equations yields, provided that it is consistent, the set of all invariant manifolds of the lowest dimension. Invariant manifolds of higher dimensions can be obtained by equating, consecutively, to zero, all the minors of higher order.

It can easily be inferred from the structure of M that any of its minors of the order $d > n - h$ (provided that it is not identically equal to zero) is equal to the corresponding (by the Laplaçe rule) $[d - (n - h)]$-th order minor of the matrix

$$\| a_{ji}^{\varepsilon},\ a_{\rho i}^{\varepsilon} \| = \| a_{\gamma i}^{\varepsilon} \| \qquad (\gamma = 1, \ldots, m) \tag{5.1}$$

Thus the above procedure of determination of invariant manifolds can be applied directly to the matrix (5.1) of the left-hand side of the defining equations (3.10). This, together with Theorem 4.1, implies that the manifolds on which the defining equations degenerate, will be invariant manifolds of the group defined by these equations.

2) Nonsingular invariant manifolds. If the general rank q of the matrix M is less than $n + m - h$, then, obviously, the overall rank r of the matrix (5.1) will be less than m. In this case the group G^* and consequently G, will admit $m - r$ invariants $\psi_1 = \text{const}, \ldots, \psi_{m-r} = \text{const}$, and a set of regular functions $\lambda_1^{(s)}, \ldots, \lambda_m^{(s)}$ will exist such, that $m - r$ independent relations $\lambda_\gamma^{(s)} a_{\gamma i}^{\varepsilon} = 0$, $s = 1, \ldots, m - r$ will hold at any point of Γ'. Functions $\lambda_\gamma^{(s)}$ can easily be found. The assumption that solutions of (3.10) continuous in Γ exist, implies that $m - r$ relations of the type

$$\lambda_j^{(s)}\xi_j + \lambda_j^{(s)}\alpha_{j\omega}\xi_\omega = \lambda_i^{(s)}\xi_i = 0$$

should hold. Conditions $X\psi_\nu = 0$, $\nu = 1, \ldots, m - r$ imply that the Pfaffian system

$$\lambda_1^{(1)}\,dx_1+\cdots+\lambda_n^{(1)}dx_n=0$$
$$\cdots\cdots\cdots\cdots$$
$$\lambda_1^{(m-r)}dx_1+\cdots+\lambda_n^{(m-r)}dx_n=0$$

is consistent and admits the invariants in question as its solutions. The above argument also applies to the invariants of a group induced by G on its invariant manifolds.

Example. We shall consider the defining equation of the group G_1 admitted by the differential equation

$$\frac{\partial u}{\partial x_1}=\left(\frac{\partial u}{\partial x_2}\right)^2$$

$$\frac{\partial\zeta}{\partial x_1}=0,\qquad 2\frac{\partial\zeta}{\partial x_2}+\frac{\partial\xi_2}{\partial x_1}=0,\qquad \frac{\partial\zeta}{\partial u}-2\frac{\partial\xi_2}{\partial x_2}+\frac{\partial\xi_1}{\partial x_1}=0$$
$$2\frac{\partial\xi_1}{\partial x_2}+\frac{\partial\xi_2}{\partial u}=0,\qquad \frac{\partial\xi_1}{\partial u}=0 \tag{5.2}$$

$$x_1'=x_1+\tau\xi_1,\qquad x_2'=x_2+\tau\xi_2,\qquad u'=u+\tau\zeta$$

Since these equations are homogeneous $(h=0)$, Theorem 4.1 implies that the group G_1 defined by them, is locally transitive at all points.

Let us supplement (5.2) with

$$x_1\frac{\partial\xi_1}{\partial x_1}+x_2\frac{\partial\xi_1}{\partial x_2}+u\frac{\partial\xi_1}{\partial u}=m\xi_1,\qquad x_1\frac{\partial\xi_2}{\partial x_1}+x_2\frac{\partial\xi_2}{\partial x_2}+u\frac{\partial\xi_2}{\partial u}=m\xi_2$$
$$x_1\frac{\partial\xi}{\partial x_1}+x_2\frac{\partial\zeta}{\partial x_2}+u\frac{\partial\zeta}{\partial n}=m\zeta \tag{5.3}$$

where m is a natural number.

It can easily be verified that the system (5.2) supplemented with (5.3) will again be a system of defining equations and, that it will be passive. Eqs. (5.3) define a homogeneous subgroup G of G_1. The matrix $\| a^{\xi}_{\gamma l}\|$ of the left-hand sides of (5.2) and (5.3) has the form

$$\begin{Vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0\\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1\\
0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
x_1 & x_2 & u & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & x_1 & x_2 & u & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & u
\end{Vmatrix}$$

Some of the fifth order minors of this matrix nowhere become equal to zero. All sixth and seventh order minors become zero at the point $x_1=x_2=u=0$ and therefore constitute a zero-dimensional invariant manifold I_0 of the group G. All eighth order minors become zero on the surface

$$4x_1u+x_2^2=0 \tag{5.4}$$

which is, in this case, a two-dimensional invariant manifold I_2 of the group G.

Since the overall rank of the matrix $r=m=8$, the group G does not admit any invariants. At the general points of the surface I_2, the rank of the matrix is reduced by one, consequently the group induced on I_2 by G is transitive.

We note that the quantity u defined by (5.4) as a function of the independent variables x_1 and x_2 is, as expected, a solution of the parent equation

$$\frac{\partial u}{\partial x_1}=\left(\frac{\partial u}{\partial x_2}\right)^2$$

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SYNCHRONIZATION OF FINITE-DIMENSIONAL "FORCE" GENERATORS

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The problem of synchronizing almost conservative dynamic objects [1] under weak constraints is considered. The character and mechanism of action of the objects on the supporting system are defined qualitatively and are no way related to its specific form [2].

The proposed procedure for investigating synchronous states in such systems is based on the notion of the dynamic influence matrix. It is shown that qualitative definition (specification) of the character of action of the objects is a natural basis for their classification. The paper ends with an examination of the synchronization of generators of "forces" of the simplest type, i.e. of one- and two-dimensional "forces".

The results can be applied, for example, to the solution of vibration engineering problems involving the properties of several complex vibration sources operating simultaneously.

The problems of synchronous state stability have already been investigated in [3 and 4], and therefore will not be considered here.

1. The dynamic influence matrix. Let us assume that the motion of the arbitrary i th object $(i = 1\ldots, n)$ in a system is completely defined if we know the time variation of $l_i \times 1$ vector columns of its proper coordinates $q_i = (q_i^{(1)}, \ldots, q_i^{(l_i)})$ and $m_i \times 1$ vectors of the reverse influence parameters $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(m_i)})$. The physical character of the reverse influence parameters is completely determined by the specifies of the object and is unrelated to the form of the supporting system [2]. The